Multi-State Delinquency Models with Random Effects for Credit Cards

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Comparison of model objectives

Repayment state at time $t=\tau$

$\tau > s$

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Repayment state at time $t=s$

Three questions a lender tries to answer.

For any account $i$

Q1. What is the probability of moving from up to date (0) to 3 behind, any time in the first $T$ months?

Q2. What is the probability of moving from state 2 to state 3 in the next month, given that it has not been in state 3 before?

Q3. What is the probability of moving from state $h$ to state $j$ between month $s$ and month $\tau$?
Advantages over static models

• Likelihood function incorporates censored cases
• Do not have to pre-define observation window (can predict for any duration time period)
• Can incorporate time varying covariates

Example

\[ \lambda_i(t, x_i(t)) = \lambda_0(t). \exp(\beta^T x_i(t)) \]
Two State Intensity (Survivor) Models of Default

Consumer:  Banasik, Crook, Thomas (1999), Stepanova & Thomas (2002),
Djeundje & Crook (2017)

Markov Chains


Multi-state Intensity Models


            Kadam & Lenk (2008), Stefanescu (2009), Figlewski et al (2012), Koopman
            (2009), Creal et al (2014)
• Four states defined: 0, 1, 2, 3, according to number of months in arrears
Methodology: Transition Intensities

\[ \alpha_{hji}(t) = \frac{P(T_i \in [t, t + dt] | T_i \geq t)}{dt} \]

\[ \alpha_{hji}(t) = Y_{hi}(t)\alpha_{hj0}(t)\exp\{\beta_{hj}^T X_i(t)\} \]

Indicator for whether individual \( i \) was in state \( h \) at time \( \tau \)

Baseline transition intensity for state \( h \) to state \( j \) at time \( \tau \)

Vector of unknown regression coefficients

Vector of covariates for individual \( i \)

\[ p_{hjit_\alpha} = F_{hj}^L \{g_{hj}(t_\alpha) + \beta_{hj}^T x_{it_\alpha}\} \]

Continuous time

Discrete time

Multi-state models for credit cards
So far $p_{hijt}$ represents probability that case $i$ in state $h$ will jump to $j$ assuming $j$ is the only state it can jump to.

Now include competing states $i$ can move to (e.g. state 1 to 0 or 1 to 2 or 1 to 1).

2 Methods:


  Reason: computing time extremely large when estimating and including frailty using first method.
Method 2 Actuarial approach

Two innovations

• Include random effects

• Use highly flexible function for baseline: B-Splines
Actuarial Approach for Intensity models with random effects

Include random effects

Replace

$$\text{Pr}(d_{it\alpha} = 1 | \mathbf{x}_i) = p_{it\alpha} = F^L(g(t_{\alpha}) + \mathbf{x}_{it}^T \mathbf{\beta} + e_i)) \quad e_i \sim N(0, \sigma^2)$$

by

$$\text{Pr}(J_{hjit\alpha} = 1 | \mathbf{x}_i) = p_{hjit\alpha} = F^L(g_{hj}(t_{\alpha}) + \mathbf{x}_{it}^T \mathbf{\beta}_{hj} + e_{hji})) \quad e_{hji} \sim N(0, \sigma^2_{hj})$$

Notice $t$ is duration time.

Flexible base line

Use B-splines for $g_{hj}(t_{\alpha})$

$$g_{hj}(t_{\alpha}) = \sum_{t=1}^{c} B_i(t) b_{i,hj}$$

where $B_i(t)$ are B-spline functions at points $t$ and $\mathbf{b}_{hj} = (b_1, \ldots, b_c)$

Is a vector of unknown coefficients to be estimated
We wish to estimate $\beta_{hj}, b_{hj}, \sigma_{hj}$

Let

$Y = (Y_{hj,i}, (hj), i)$ where $Y_{hj,i}(t) = \begin{cases} 1 & \text{if account } i \text{ is in state } j \text{ at time } t, \text{ given it was in } h \text{ at } t - 1 \\ 0 & \text{if account } i \text{ is in state } h \text{ at time } t, \text{ given it was in } h \text{ at } t - 1 \end{cases}$

$\beta = (\beta_{hj}, (h, j))$

$u = (u_{hj,i}, (h, j), i)$

$\phi_u = \text{multivariate normal density}$

Joint likelihood of $(Y, u)$ is

$L_{(Y, u)} = (\beta, b, \sigma) = L_{(Y|u)}(\beta, u) \times \phi_u(\sigma)$  
where  
$\phi_u(\sigma) \propto \left|\sigma^{-0.5}\right| \exp\left(-\frac{1}{2} u^T \sigma^{-1} u\right)$

and

$L_{(Y|u)} = (\beta, b, \sigma) = \prod_{(h, j)} \prod_t \prod_{i \in \mathcal{R}_{hj}(t)} \left[P_{hji}(t)^{y_{hji}(t)} \times [1 - P_{hji}(t)]^{1-y_{hji}(t)}\right]$  

$\mathcal{R}_{hj(t)} = \text{risk set for transitions from } h \text{ to } j \text{ at } t.$

We wish to maximise the marginal likelihood

$$\max_{\beta, b} L_{(Y)}(\beta, b, \sigma) = \max_{\beta, b} \int L_{(Y, u)} = (\beta, b, \sigma) du$$
Competing probabilities

\[ \tilde{P}_{hji}(t) = p_{hji}(t) \times \left\{ 1 - \frac{1}{2} \sum_{k \neq j \text{ where } (h,k) \in \varphi} p_{hji}(t) + \frac{1}{3} \sum_{k \neq r \neq j \text{ where } (h,k) \in \varphi} p_{hki}(t) p_{hri}(t) - \frac{1}{4} \sum_{\text{where } (h,k) \in \varphi \text{ and } (h,r) \in \varphi \text{ and } (h,s) \in \varphi} p_{hki}(t) p_{hri}(t) p_{hsi}(t) + \ldots \right\} \]

\[ \tilde{P}_i(t) = \begin{bmatrix} (1 - \tilde{p}_{01,i}(t)) & \tilde{p}_{01,i}(t) & 0 & 0 \\ \tilde{p}_{10,i}(t) & (1 - \tilde{p}_{10,i}(t) - \tilde{p}_{12,i}(t)) & \tilde{p}_{12,i}(t) & 0 \\ \tilde{p}_{20,i}(t) & \tilde{p}_{21,i}(t) & 1 - \tilde{p}_{20,i}(t) - \tilde{p}_{21,i}(t) - \tilde{p}_{23,i}(t) & \tilde{p}_{23,i}(t) \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Probs that account \( i \) in state \( \delta_i(t_1) \) at time \( t_1 \) will be in state 0, 1, 2, or 3 at time \( t_2 \) are elements in following vector, \( \mu(t_2) \), given by the matrix product

\[ \mu(t_2) = \left[ 1_{\{\delta_i(t_1)=0\}}, 1_{\{\delta_i(t_1)=1\}}, 1_{\{\delta_i(t_1)=2\}}, 1_{\{\delta_i(t_1)=3\} \right] \tilde{P}_i(t_1, t_2) \]

where 1 denotes indicator operator

\[ \tilde{P}_i(t_1, t_2) \text{ represents the cumulative transition prob matrix given by } \tilde{P}_i(t_1, t_2) = \prod_{t=t_1+1}^{t_2} \tilde{P}_i(t) \]
Data

Credit card accounts tracked monthly

- Application variables
- Behavioural indicators e.g. spending amount, repayment amount (if any)
- Macroeconomic variables
Results
Smoothed baseline functions

Multi-state models for credit cards
<table>
<thead>
<tr>
<th>Variable</th>
<th>Delinquency type</th>
<th>Recovery type</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T 01</td>
<td>T 12</td>
</tr>
<tr>
<td>Number of cards</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>Landline Y/N</td>
<td>n</td>
<td>-</td>
</tr>
<tr>
<td>Time at address</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>Time with bank</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Time with bank, missing</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>Income, ln</td>
<td>n</td>
<td>+</td>
</tr>
<tr>
<td>Income, missing</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Housing type (categorical)</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Age group (categorical)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Employment status (categorical)</td>
<td>Mixed</td>
<td>Mixed</td>
</tr>
<tr>
<td>Credit limit, ln, lag6</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Repayment amt, ln, lag6</td>
<td>+</td>
<td>n</td>
</tr>
<tr>
<td>Proportion of credit drawn, lag6</td>
<td>+</td>
<td>n</td>
</tr>
<tr>
<td>Rate of total jumps, lag6</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Improvement in state from 3 months previous, lag6</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>RPI, NSA, lag6</td>
<td>+</td>
<td>n</td>
</tr>
<tr>
<td>AWE, NSA, lag6</td>
<td>+</td>
<td>n</td>
</tr>
<tr>
<td>FTSE, NSA, lag6</td>
<td>+</td>
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</tr>
<tr>
<td>Unemployment rate, SA, lag6</td>
<td>n</td>
<td>-</td>
</tr>
<tr>
<td>IOP, NSA, lag6</td>
<td>-</td>
<td>n</td>
</tr>
<tr>
<td>HPI, SA, lag6</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>Consumer confidence, NSA, lag6</td>
<td>n</td>
<td>n</td>
</tr>
<tr>
<td>Credit card IR, NSA, lag6</td>
<td>-</td>
<td>n</td>
</tr>
<tr>
<td>Mortgage loan IR, NSA, lag6</td>
<td>+</td>
<td>n</td>
</tr>
<tr>
<td>Total credit outstanding, ln, NSA, lag6</td>
<td>-</td>
<td>n</td>
</tr>
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Multi-state models for credit cards
Diagnostics

<table>
<thead>
<tr>
<th>Jump</th>
<th>01</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>21</th>
<th>23</th>
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<tr>
<td>$\sigma^2_{hj}$</td>
<td>1.14</td>
<td>1.59</td>
<td>0.98</td>
<td>2.36</td>
<td>1.78</td>
<td>1.94</td>
</tr>
</tbody>
</table>

Deviance residuals all within 2 sds of mean for (virtually) all time periods for all transitions
Use test set (accounts opened 2008 onwards)

For a given value of a covariate eg age, for each account for each transition type \((h, j) \in \mathcal{G}\)

- simulate values of a random deviate \(u_{hji}\) from \(\mathcal{N}(0, \sigma_{hj}^2)\) where \(\sigma_{hj}\) are from the estimd eqtns
- add to linear predictor of \(p^{(hj)}_{i}(t)\) to get \(\hat{p}^{(hj)}_{i}(t)\)
- for each \(t\) take average over \(i\)
- Repeat for each value of the covariate
Multi-state models for credit cards

Predicting transition probabilities by employment category

Retired, Unemployed (grey) low chance of becoming delinquent, high chance of recovering.

Employed (dk blue) average chance of missing one, high chance of missing further.

Low chance of recovering from 1.

Self Empl (orange) high chance of missing one, average chance of missing more, low chance of moving to 3, high chance of recovering.

Students (yellow) low chance of missing one, if do high chance of moving to 2 and from 2 to 3.

9/7/2017
Predicting transition probabilities by age at application

Older (brown) less likely to miss 1, or 2, most likely to recover

Youngest (light blue) less likely to miss 1, if do least likely to recover and more likely to move to 2

Young (yellow) more to miss 1 then 2 then 3. Least likely to recover from 1 or 2.
Cumulative transition probability matrix, $P(6,12)$, by employment type for typical account opened in January 2009

<table>
<thead>
<tr>
<th>Employment</th>
<th>From state</th>
<th>To state</th>
<th></th>
<th></th>
<th></th>
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<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td></td>
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<tr>
<td>Employment A</td>
<td>0.9020</td>
<td>0.0634</td>
<td>0.0167</td>
<td>0.0179</td>
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<tr>
<td>Employment B</td>
<td>0.9040</td>
<td>0.0595</td>
<td>0.0198</td>
<td>0.0167</td>
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<td></td>
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<tr>
<td>Employment C</td>
<td>0.9698</td>
<td>0.0235</td>
<td>0.0034</td>
<td>0.0032</td>
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<tr>
<td>Employment D</td>
<td>0.9351</td>
<td>0.0287</td>
<td>0.0110</td>
<td>0.0252</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Employment E</td>
<td>0.7021</td>
<td>0.0241</td>
<td>0.0113</td>
<td>0.2625</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\( \tilde{P}_i(t_1, t_2) \) represents the cumulative transition prob matrix given by

\[
\tilde{P}_i(t_1, t_2) = \prod_{t=t_1+1}^{t_2} \tilde{P}_i(t)
\]

Let \( \hat{P}_{ik0}, \hat{P}_{ik1}, \hat{P}_{ik2}, \hat{P}_{ik3} \) denote predicted competing probabilities that an account will be in state 0, 1, 2, 3 at time \( t_2 \) given it was in state \( k \) at \( t_1 \) that is the elements in row \( k \) in \( \tilde{P}_i(t_1, t_2) \).

To predict state need to compare predicted probabilities with cut off.

At \( t_2 \) we predict account \( i \) will be in state \( j \) such that

\[
\hat{P}_{kj} - c_{kj} = \max \{ p_{k0} - c_{k0}, p_{k1} - c_{k1}, p_{k2} - c_{k2}, p_{k3} - c_{k3} \}
\]

where

\( c_{k0}, c_{k1}, c_{k2}, c_{k3} \) as the multidimensional maximisers of \( f_k \) where

\[
f_k(a_0, a_1, a_2, a_3) = \frac{1}{N_k(t_1)} \sum_{\delta_i(t_1)=k, \delta_i(t_2)=t_2} 1_{\{\delta_i(t_2|a_0,a_1,a_2,a_3)=\delta_i,t_2\}}
\]
Standardised discrepancy

\[
\frac{\hat{p}_{kj} - c_{kj}}{\hat{s}_{kj}} = \max\left\{ \frac{\hat{p}_{k0} - c_{k0}}{\hat{s}_{k0}}, \frac{\hat{p}_{k1} - c_{k1}}{\hat{s}_{k1}}, \frac{\hat{p}_{k2} - c_{k2}}{\hat{s}_{k2}}, \frac{\hat{p}_{k3} - c_{k3}}{\hat{s}_{k3}} \right\}
\]

Cumulative discrepancy

\[
\frac{\hat{p}_{kj} - c_{kj}}{c_{kj}} = \max\left\{ \frac{\hat{p}_{k0} - c_{k0}}{c_{k0}}, \frac{\hat{p}_{k1} - c_{k1}}{c_{k1}}, \frac{\hat{p}_{k2} - c_{k2}}{c_{k2}}, \frac{\hat{p}_{k3} - c_{k3}}{c_{k3}} \right\}
\]
## Predictive performance

### Prediction accuracy at time 12, given state at time 6 (%)

<table>
<thead>
<tr>
<th>State at time 6</th>
<th>No random effects</th>
<th>With random effects</th>
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<tr>
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<td>discrepancy</td>
<td>stand. discrepancy</td>
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<tr>
<td>0</td>
<td>89</td>
<td>90</td>
</tr>
<tr>
<td>1</td>
<td>72</td>
<td>73</td>
</tr>
<tr>
<td>2</td>
<td>63</td>
<td>62</td>
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Survival models are the **second** generation of credit scoring models.

Multistate intensity models are the **third** generation.

Multistate intensity models yield predictions of transition probabilities between delinquency states. They are useful for IFRS9, the prediction of provisions and for economic capital prediction.

Including frailty in multistate intensity models:

- substantially increases estimation times so need to move from continuous time models to discrete time models
- alters coefficients noticeably and renders many covariates measured for each account insignificant
- leads macroeconomic variables to be significant
- Increases prediction accuracy.

Use of B-splines gives more accurate predictions than more restrictive baseline functions.